Logarithmic corrections to gap scaling in random-bond Ising strips

1997 J. Phys. A: Math. Gen. 30 L443

(http://iopscience.iop.org/0305-4470/30/14/001)

J. Phys. A: Math. Gen. **30** (1997) L443–L447. Printed in the UK PII: S0305-4470(97)83212-X

LETTER TO THE EDITOR

Logarithmic corrections to gap scaling in random-bond Ising strips

S L A de Queiroz†

Instituto de Física, UFF, Avenida Litorânea s/n, Campus da Praia Vermelha, 24210-340 Niterói RJ, Brazil

Received 8 April 1997, in final form 13 May 1997

Abstract. Numerical results for the first gap of the Lyapunov spectrum of the self-dual random-bond Ising model on strips are analysed. It is shown that finite-width corrections can be fitted very well by an inverse logarithmic form, predicted to hold when the Hamiltonian contains a marginal operator.

It is widely believed that the critical behaviour of the two-dimensional Ising model is only slightly modified by the introduction of non-frustrated disorder [1, 2]. Such changes are given by logarithmic corrections to pure-system power-law singularities. In terms of a field-theoretical (or renormalization-group) description, disorder is said to be a marginally irrelevant operator [3, 4]. Specific forms have been proposed, and numerically tested, for the corrections to bulk quantities as specific heat, magnetization and initial susceptibility [1]. The overall picture emerging from such analyses tends to confirm predictions of log-corrected pure-Ising behaviour. Early proposals of drastic alterations in the values of critical indices [5] have thus been essentially discarded. However, recent results [6, 7] have appeared, according to which critical indices would vary with disorder, but so as to keep the ratio γ/ν constant at the pure system's value (the so-called *weak universality* scenario [8]). In order to solve this controversy, it is important to undertake independent tests of several aspects of the problem. In the present work we examine the finite-width corrections to the first gap of the Lyapunov spectrum of the disordered Ising model on strips. This gap is related to the typical (as opposed to averaged) behaviour of spin–spin correlation functions, as explained below.

While the average of a random quantity $\mathcal{Q}$ is simply its arithmetic average over independent realizations $\mathcal{Q}_i$, $\overline{\mathcal{Q}} = (1/N)\sum_{i=1}^{N}\mathcal{Q}_i$, its typical (in the sense of most probable) value is expected to agree with the geometrical average: $\mathcal{Q}_{\rm typ} = \mathcal{Q}_{\rm m.p.} = \exp(\overline{\ln \mathcal{Q}})$ [9–12]. Depending on the underlying probability distribution, $\overline{\mathcal{Q}}$ and $\mathcal{Q}_{\rm typ}$ may differ appreciably, as is the case when one considers correlation functions [9, 11].

It has been predicted, on the basis of field-theoretical arguments [3], that in a bulk system (as opposed to the strip geometry used here) the *typical* decay of spin–spin correlation functions at criticality on a fixed sample is given by

$$\langle\sigma_0\sigma_R\rangle_{\rm typ} \propto R^{-1/4}(\Delta \ln R)^{-1/8} \quad \text{(fixed sample)} \tag{1}$$

† E-mail address: sldq@if.uff.br

for $\ln(\Delta \ln R)$ large, where Δ is proportional to the intensity of disorder. This way, pure-system behaviour (power-law decay against distance, $\langle \sigma_0 \sigma_R \rangle \sim R^{-\eta}$ with $\eta = 1/4$) acquires a logarithmic correction. On the other hand, if one considers the *average*, over many samples, of the correlation function, logarithmic corrections are expected to be washed away [3] resulting in a simple power-law dependence:

$$\overline{\langle \sigma_0 \sigma_R \rangle} \propto R^{-1/4} \quad \text{(average over samples)}. \tag{2}$$

The distinction between typical and average correlation decay was not explicitly discussed in the early field-theoretical treatment [5] which predicted $\langle \sigma_0 \sigma_R \rangle \propto \mathrm{e}^{-A(\ln R)^2}$, a sort of behaviour for which no evidence has been found in subsequent investigations [1]. Recent numerical work claiming weak universality to hold [6, 7] does not address the issue either, although it is easy to see that the procedures used in those calculations pick out averaged correlations, as they rely respectively on variants of the fluctuation-dissipation theorem [6] or on explicit averaging [7].

In [7], an analysis of phenomenological renormalization estimates of the correlation-length exponent ν seems to point against logarithmic corrections to its pure-system value, and in favour of a disorder-dependent exponent. However, that is based on data for very narrow strips ($L \leqslant 8$) and mostly relies on trends apparently followed at weak disorder. Although one series of moderately-strong disorder results is also exhibited, no attempt is made to fit either set to the form predicted by theory, which crucially includes a disorder-dependent crossover length [1]. Indeed, a systematic treatment of the averaged correlation lengths at the exact critical point, spanning a wide range of disorder and watching for the disorder-dependent crossover mentioned above, eventually uncovers the expected logarithmic terms [2]. Turning back to the exponent η, recall that, as far as dominant behaviour is concerned, both weak- and strong-universality concur in predicting $\eta = 1/4$. Thus, our strategy here will be to analyse the *subdominant* terms. Our goal is to show that the finite-size corrections to the typical ratio of decay of correlations behave consistently with what is expected when a marginal operator is present (see later).

In numerical simulations, one considers finite lattices ($L \times L$) or finite-width strips ($L \times N$, $N \to \infty$) and sets the temperature at the critical point of the corresponding two-dimensional system. For the nearest-neighbour random-bond Ising model on a square lattice, with a binary distribution of ferromagnetic interactions

$$P(J_{ij}) = \tfrac{1}{2}(\delta(J_{ij} - J_0) + \delta(J_{ij} - rJ_0)) \qquad 0 \leqslant r \leqslant 1 \tag{3}$$

the critical temperature $\beta_{\mathrm{c}} = 1/k_{\mathrm{B}}T_{\mathrm{c}}$ is exactly known [13, 14] from self-duality as a function of r through:

$$\sinh(2\beta_{\mathrm{c}}J_0)\sinh(2\beta_{\mathrm{c}}rJ_0) = 1. \tag{4}$$

From Monte Carlo work on $L \times L$ random-bond systems [15], it has been found that the average correlation function at criticality is numerically very close to the exactly known [16] value for a pure system of the same size at its own critical point, thus providing evidence in favour of (2). Similar conclusions have been drawn for the corresponding quantities evaluated on strips [17, 18], where the exact critical correlation functions for the pure Ising model are known from conformal invariance [19].

Here we provide a test of the consequences of (1), when correlations are considered on a strip geometry. In this case, contrary to that of (2), no exact finite-L results are available for comparison; thus one must resort to finite-size scaling concepts [20], in order to unravel signs of the expected bulk behaviour from trends followed by finite-system data as $L \to \infty$.

The procedure used is as follows. It is known that the typical, or most probable, spin–spin correlation function on a strip decays as

$$\langle\sigma_0\sigma_R\rangle_{\rm typ} \propto \exp(-R/\xi_{\rm typ}) \qquad \xi_{\rm typ}^{-1} = \Lambda_L^0 - \Lambda_L^1 \tag{5}$$

where Λ_L^0 and Λ_L^1 are the two largest Lyapunov exponents of the (random) transfer matrix for a strip of width L [10–12]. On the other hand, conformal invariance predicts [21] that, when the Hamiltonian of a homogeneous two-dimensional system contains a marginal operator, the spectrum of eigenvalues E_n of the transfer matrix on a strip is such that

$$E_n - E_0 = (2\pi/L)(x_n + d_n/\ln L) + \cdots \tag{6}$$

where x_n is the corresponding scaling dimension, d_n is an n-dependent constant and periodic boundary conditions are used across the strip. Since (i) disorder in the two-dimensional Ising model is believed to be a marginal operator, (ii) Lyapunov exponents of transfer matrices in random systems are the counterparts of eigenvalues in homogeneous ones, and (iii) numerical evidence shows that conformal-invariance results derived for uniform systems may be extended to random cases, provided that suitable averages are taken [2, 17], we shall examine sequences of estimates $\Lambda_L^0 - \Lambda_L^1$ for varying L and try to fit them to (6) with $n = 1$ and $2x_1 = \eta = 1/4$ [19].

We have used strips of width $L \leqslant 13$ sites, and length $N = 10^5$ for $L = 2$–11 and 5×10^4 for $L = 12$ and 13. Two values were taken for the disorder parameter r of (3): $r = 0.5$ and 0.01. Details of the calculation are given in [17], where the data used here are displayed as well, in plots of $(L/\pi)(\Lambda_L^0 - \Lambda_L^1)$ against $1/L^2$. The latter variable was used because it is exactly known [22] that, for strips of pure Ising spins, the leading corrections to the L^{-1} dependence of ξ^{-1} are proportional to L^{-2}. Our purpose there was to show that the correlation length $\xi_{\rm ave}$ coming from direct average of correlation functions, $\overline{\langle\sigma_0\sigma_R\rangle} \sim \exp(-R/\xi_{\rm ave})$, indeed scales as its pure-system counterpart, while $\xi_{\rm typ}$ does not. At the time we did not investigate the behaviour of $\xi_{\rm typ}$ in detail, though it was noticed that for strong disorder the curvature of plots of $L/\pi\xi_{\rm typ}$ against $L^{-\phi}$ only became smaller for ϕ close to zero.

In figures 1(a) and 1(b) we show $L/\pi\xi_{\rm typ} \equiv (L/\pi)(\Lambda_L^0 - \Lambda_L^1)$ against $1/\ln L$, respectively for $r = 0.5$ and 0.01. For weak disorder $r = 0.5$ the fit to (6) is very good, with $2x_1 = \eta = 1/4$. While for $r = 0.01$ strong fluctuations increase the amount of scatter, the overall picture is still consistent with (6). Table 1 shows results from least-squares fits of data in the range $L = 6$–13. This interval was chosen in order to minimize the accumulated standard deviation χ^2 per degree of freedom [17]. The extrapolated η is expected to be universal as long as $r \neq 0$; our error bar for $r = 0.01$ indeed includes $\eta = 1/4$, though admittedly at the edge. On the other hand, d_1 clearly changes, increasing with disorder. The situation differs from that of the q-state Potts model with $q = 4$ [21, 23, 24] where there is no continuously tunable parameter, as marginality depends on $q - 4$ being strictly zero.

Table 1. Results from least-squares fits to (6), for $L = 6$–13.

r	η	d_1	χ^2
0.5	0.250 ± 0.006	0.018 ± 0.013	0.021
0.01	0.29 ± 0.04	0.38 ± 0.09	0.17
Expected	1/4	—	—

As an additional check on the ideas developed previously, we examined data for the probability distribution of critical spin–spin correlation functions [18] for $r = 0.25$. For

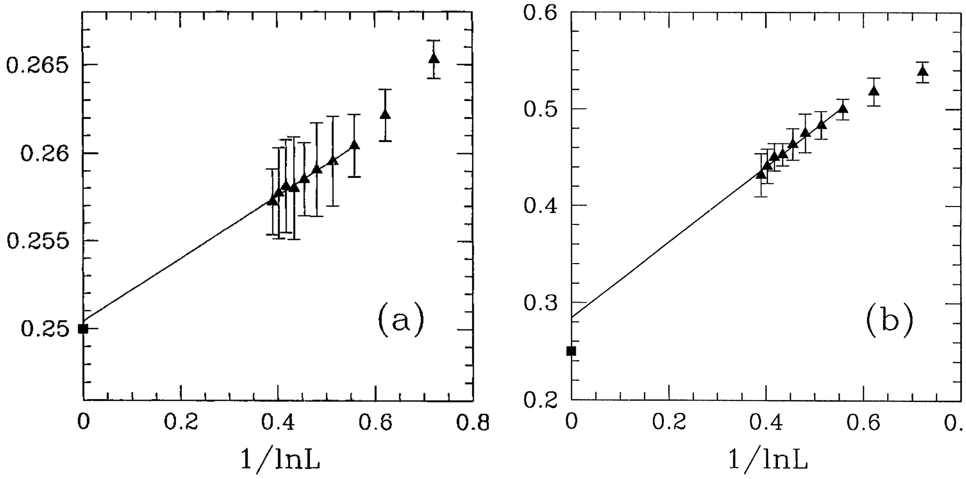

Figure 1. $\eta \equiv L/\pi\xi_{\rm typ}$, with $\xi_{\rm typ}^{-1} = \Lambda_L^0 - \Lambda_L^1$ of (6), against $1/\ln L$. The square on the vertical axis is at the pure system value $\eta = 1/4$. Straight lines are least-square fits for $L = 6$–13. (a) $r = 0.5$; (b) $r = 0.01$.

spin–spin distances $R = 5$ and 20, and strip widths $L = 3, \ldots, 13$ we picked the averages $\overline{\ln G(R)} \equiv \overline{\ln(\langle\sigma_0\sigma_R\rangle)}$. The quantity $\exp\overline{\ln G(R)}$ is expected [11, 12] to scale as $\langle\sigma_0\sigma_R\rangle_{\rm typ}$. For each L the slope of a two-point semi-log plot of $\exp\overline{\ln G(R)}$ gave an approximate value for $1/\xi_{\rm typ}$. From a plot of $L/\pi\xi_{\rm typ}$ against $1/\ln L$ one gets $\eta \simeq 0.26$, $d_1 \simeq 0.03$, consistent with the above values derived directly from Lyapunov exponents, and with the assumption that d_1 varies continuously (and monotonically) against disorder.

We have analysed numerical estimates for the first gap of the Lyapunov spectrum of the self-dual random-bond Ising model on strips. We have shown that finite-width corrections can be fitted very well by an inverse logarithmic form, predicted to hold when the Hamiltonian contains a marginal operator. The present results contribute, albeit indirectly, to the growing body of evidence favouring strong universality (that is, pure-system behaviour with logarithmic corrections) in the two-dimensional random Ising model [1–4]. This is to be contrasted with recent work [6, 7], according to which critical indices would vary with disorder, but so as to keep the ratio γ/ν constant at the pure system's value (the so-called *weak universality* scenario [8]).

The author thanks CNPq, FINEP and CAPES for financial support.

References

[1] Selke W, Shchur L N and Talapov A L 1994 *Annual Reviews of Computational Physics* vol 1, ed D Stauffer (Singapore: World Scientific) p 17
[2] Aarão Reis F D A, de Queiroz S L A and dos Santos R R 1996 *Phys. Rev.* B **54** R9616
[3] Ludwig A W W 1990 *Nucl. Phys.* B **330** 639
[4] Shalaev B N 1994 *Phys. Rep.* **237** 129
[5] Dotsenko Vik S and Dotsenko Vl S 1982 *J. Phys. C: Solid State Phys.* **15** 495
[6] Kim J-K and Patrascioiu A 1994 *Phys. Rev. Lett.* **72** 2785
[7] Kühn R 1994 *Phys. Rev. Lett.* **73** 2268
[8] Suzuki M 1974 *Prog. Theor. Phys.* **51** 1992
[9] Derrida B and Hilhorst H 1981 *J. Phys. C: Solid State Phys.* **14** L539
[10] Derrida B 1984 *Phys. Rep.* **103** 29

[11] Crisanti A, Nicolis S, Paladin G and Vulpiani A 1990 *J. Phys. A: Math. Gen.* **23** 3083

[12] Crisanti A, Paladin G and Vulpiani A 1993 *Products of Random Matrices in Statistical Physics (Springer Series in Solid State Sciences)* vol 104, ed H K Lotsch (Berlin: Springer) p 59

[13] Fisch R 1978 *J. Stat. Phys.* **18** 111

[14] Kinzel W and Domany E 1981 *Phys. Rev.* B **23** 3421

[15] Talapov A L and Shchur L N 1994 *Europhys. Lett.* **27** 193

[16] Wu T T, McCoy B M, Tracy C A and Barouch E 1976 *Phys. Rev.* B **13** 376

[17] de Queiroz S L A 1995 *Phys. Rev.* E **51** 1030

[18] de Queiroz S L A and Stinchcombe R B 1996 *Phys. Rev.* E **54** 190

[19] Cardy J L 1987 *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J L Lebowitz (London: Academic) p 55

[20] Barber M N 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (London: Academic) p 145

[21] Cardy J L 1986 *J. Phys. A: Math. Gen.* **19** L1093

[22] Derrida B and de Seze L 1982 *J. Physique* **43** 475

[23] Blöte H W J and Nightingale M P 1982 *Physica* A **112** 405

[24] Nightingale M P and Blöte H W J 1983 *J. Phys. A: Math. Gen.* **16** L657